An upper bound for characteristic functions of lattice distributions with applications to survival probabilities of quantum states

2007 J. Phys. A: Math. Theor. 40 131

(http://iopscience.iop.org/1751-8121/40/1/007)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 131–137 doi:10.1088/1751-8113/40/1/007

An upper bound for characteristic functions of lattice distributions with applications to survival probabilities of quantum states

Zhengmin Zhang

School of Mathematics and Statistics, Carleton University, Ottawa, ON, K1S 5B6, Canada

E-mail: zhangzm@math.carleton.ca

Received 3 July 2006, in final form 14 November 2006
Published 6 December 2006
Online at stacks.iop.org/JPhysA/40/131

Abstract
An inequality which sets an upper bound for characteristic functions (Fourier transform of probabilities) of lattice distributions is established in terms of a discrete version of the Fisher information. Its physical implications for the survival probabilities and lifetime of quantum states are illustrated.

PACS numbers: 02.30.Nw, 02.50.Cw, 03.65.Ta

1. Introduction

Let F be any probability distribution function, that is, $F : R \to [0, 1]$ is non-decreasing, right continuous, with $\lim_{t\to-\infty} F(t) = 0$, $\lim_{t\to\infty} F(t) = 1$. Let

$$\phi(t) = \int_R e^{itx}\, dF(x), \qquad t \in R$$

be its characteristic function [2, 6]. In the terminology of harmonic analysis, $\phi(-t)$ is precisely the Fourier transform of the distribution F. In quantum mechanics, $\phi(-t)$ has the physical interpretation as the survival amplitude (decay amplitude) of a quantum state [3].

The characteristic function $\phi(t)$ not only encodes all information about the probability distribution F, but also synthesizes it in a most convenient way for many purposes such as spectral analysis. The behaviours of $\phi(t)$ are the focus of many studies for both theoretical and practical reasons. In particular, they are fundamental objects in probability theory [2, 6], and when the characteristic functions are interpreted as the survival amplitudes of quantum states they are relevant for quantifying time–energy uncertainty relations and decaying rate of quantum states [3, 7–14].

According to Feller [2] (p 501, lemma 4), there exists only the following three possibilities for $\phi(t)$:

(1) $|\phi(t)| = 1$ for all $t \in R$.

(2) $|\phi(t)| < 1$ for all $t \in R, t \neq 0$.
(3) $|\phi(T)| = 1$ for a positive T and $|\phi(t)| < 1$ for $0 < t < T$.

Case (1) is trivial since in this case $\phi(t) = \mathrm{e}^{\mathrm{i}at}$ for some $a \in R$ and the distribution F is concentrated at the point a. Case (2) arises often (but not necessarily) when F is absolutely continuous and thus possesses a probability density. In case (3), $\phi(t)$ has period T and there exists a real number b such that the probability distribution F is supported on the lattice $\{b + kh : k \in Z\}$ with $h = \frac{2\pi}{T}$ and Z denoting the set of all integers.

In this paper, we are interested in seeking upper bounds for $|\phi(t)|$ in case (3), i.e., the case when F is a lattice distribution. In this situation, F is specified by a probability vector $p = \{p_k, k \in Z\}$, which is a distribution supported on the lattice point $\{b + kh : k \in Z\}$. To simplify matters without loss of generality, we may further assume $b = 0, h = 1$ (the general cases can be easily transformed to this 'standardized' case). For later purpose, it is convenient to introduce a random variable X taking values on the integer lattice Z such that the probability that X equals k is p_k, that is,

$$P(X = k) = p_k, \qquad k \in Z.$$

The characteristic function of X or, equivalently, of $p = \{p_k : k \in Z\}$ is

$$\phi(t) = \sum_{k\in Z} \mathrm{e}^{\mathrm{i}tk} p_k, \qquad t \in R.$$

In the study of local limit theorems of lattice distributions, it is required to estimate upper bounds for $|\phi(t)|$, and to estimate the integral [1, 4]

$$\int_{\delta\leqslant|t|\leqslant\pi} |\phi(t)|^n \,\mathrm{d}t, \qquad 0 < \delta < \pi.$$

This is one motivation for our interest in pursing upper bounds for $|\phi(t)|$. Another motivation lies in the fact that in quantum mechanics upper bounds for $|\phi(t)|$ give useful information about the dynamics of quantum states.

We will establish some new inequalities concerning $\phi(t)$ in terms of a discrete version of the Fisher information of the corresponding probability distribution. In particular, we obtain an upper bound for $|\phi(t)|$ when F is a lattice distribution. But first, to gain some intuition and feeling about controlling $\phi(t)$, let us briefly review some lower and upper bound estimates for characteristic functions which have interesting implications for quantum dynamics.

First, there are various lower bounds for $|\phi(t)|$ such as the celebrated Mandelstam–Tamm inequality [13],

$$|\phi(t)| \geqslant \cos(t\sigma), \qquad \forall\ |t| \leqslant \frac{\pi}{2\sigma}, \tag{1}$$

and

$$|\phi(t)| \geqslant 1 - \tfrac{1}{2}\sigma^2|t|^2, \qquad \forall\ t \in R. \tag{2}$$

Here, σ^2 is the variance of the probability distribution. If the variance is not necessarily finite, the above two inequalities are not applicable, but we still have for any $0 \leqslant \beta \leqslant 2$ [9],

$$|\phi(t)| \geqslant 1 - c_\beta M_\beta |t|^\beta, \qquad \forall\ t \in R.$$

Here, $M_\beta = \int_R |x|^\beta \,\mathrm{d}F(x)$ is the βth absolute moment of F and c_β is a positive constant (with $c_2 = 1/2$). Another similar bound is given in [11]. All the above inequalities have direct consequences for quantifying the time–energy uncertainty relations from the evolution speed perspective [8, 9, 13].

Second, concerning upper bounds for $|\phi(t)|$, a typical result is that if there exist constants a and b such that $|\phi(t)| \leqslant a < 1$ whenever $|t| \geqslant b$, then ([6], p 61)

$$|\phi(t)| \leqslant 1 - \frac{1-a^2}{8b^2}t^2, \qquad \forall\ |t| < b.$$

Clearly, this inequality is useless when F is a lattice distribution on the integer lattice, since then the condition $|\phi(t)| < 1$ cannot be satisfied for any a, in fact, $\phi(2k\pi) = 1$ for all $k \in Z$.

Some other upper bounds which have interesting implications for quantum dynamics are [2]

$$|\phi(t)|^2 \leqslant \tfrac{1}{2}(1 + |\phi(2t)|) \tag{3}$$

and [12]

$$|\phi(t)| \leqslant |\phi(nt)|\frac{1}{n}\sin\left(\frac{\pi}{2n}\right) + \cos\left(\frac{\pi}{2n}\right), \qquad n = 1, 2, \ldots. \tag{4}$$

Inequality (3) appeared in Feller [2], p 527. Inequality (4) can be derived readily from a similar result in Luo and Zhang [12]. These inequalities are self-referencing in the sense that they set an upper bound for $|\phi|$ at some point t in terms of the value of $|\phi|$ at some other point.

Our main result, which will be presented in section 2, is in some sense an inequality dual to inequalities (1) and (2). It incorporates a key characteristic, namely, a discrete analogue of the Fisher information as defined in [5], of the lattice probability distribution. In section 3, we indicate some implications of our results in analysing lifetime of quantum harmonic oscillator states.

2. An upper bound for characteristic functions

In order to establish an upper bound for $|\phi(t)|$, we first prove a Cramér–Rao-type inequality in matrix form when the underlying random variable is integer valued. It is essentially a consequence of the Schwarz inequality, and may be of independent interest. In the following, for two self-adjoint matrices A and B, the matrix inequality $A \geqslant B$ means that $A - B$ is non-negative definite.

Lemma 1. *Let X be a random variable taking values on the integer lattice Z such that $P(X = k) = p_k > 0, k \in Z$. Let $\mathbf{g}(x) = (g_1(x), g_2(x), \ldots, g_n(x)) : Z \to R^n$ be a vector function defined on Z such that the variance $Var(g_j(X)) < \infty$, $j = 1, 2, \ldots, n$. Then we have the matrix inequality*

$$\mathrm{Cov}(\mathbf{g}(X)) \geqslant \frac{1}{I(p)}(\mathrm{E}(D\mathbf{g}(X))'\mathrm{E}(D\mathbf{g}(X)).$$

Here, $\mathrm{Cov}(\mathbf{g}(X))$ *is the covariance matrix of* $\mathbf{g}(X) = (g_1(X), g_2(X), \ldots, g_n(X))$*, the symbol E denotes expectation,* $D\mathbf{g}(x) = (Dg_1(x), Dg_2(x), \ldots, Dg_n(x))$ *is a row vector with*

$$Dg_j(x) = g_j(x+1) - g_j(x), \qquad x \in Z, \qquad j = 1, 2, \ldots, n$$

and the prime$'$ denotes the transpose of a vector. $I(p)$ *is defined as*

$$I(p) = \sum_{k\in Z} \frac{(p_k - p_{k-1})^2}{p_k} \tag{5}$$

which is a discrete version of the Fisher information and is assumed to be finite.

Proof. We need to show that for any $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in R^n$, it holds that

$$\mathbf{a}\mathrm{Cov}(\mathbf{g}(X))\mathbf{a}' \geqslant \frac{1}{I(p)}\mathbf{a}(\mathrm{E}(D\mathbf{g}(X))'\mathrm{E}(D\mathbf{g}(X))\mathbf{a}'. \tag{6}$$

Suppose that $\mathrm{E}(g_j(X)) = \mu_j$, $j = 1, 2, \ldots, n$, then simple manipulation shows that

$$\sum_{k\in Z}(p_k - p_{k-1})\left(\sum_j a_j(g_j(k) - \mu_j)\right) = \sum_j a_j\left(\sum_{k\in Z}(p_k - p_{k-1})(g_j(k) - \mu_j)\right)$$
$$= -\sum_j a_j \mathrm{E}(Dg_j(X)).$$

Accordingly, by the Schwarz inequality,

$$\left|\sum_j a_j \mathrm{E}(Dg_j(X))\right|^2 = \left|\sum_{k\in Z}\frac{p_k - p_{k-1}}{\sqrt{p_k}}\cdot\sqrt{p_k}\left(\sum_j a_j(g_j(k) - \mu_j)\right)\right|^2$$
$$\leqslant \sum_{k\in Z}\frac{(p_k - p_{k-1})^2}{p_k}\cdot\sum_{k\in Z}p_k\left(\sum_j a_j(g_j(k) - \mu_j)\right)^2$$
$$= I(p)\cdot \mathrm{Var}\left(\sum_j a_j g_j(X)\right).$$

Rewritten the above in matrix forms, we obtain inequality (6), which is the desired result. □

Theorem 1. *Let $p = \{p_k : k \in Z\}$ be a lattice distribution supported on the whole integer lattice Z (that is, $p_k > 0$ for all $k \in Z$ and $\sum_{k\in Z} p_k = 1$) with characteristic function $\phi(t) = \sum_{k\in Z} \mathrm{e}^{\mathrm{i}tk} p_k$, then*

$$|\phi(t)|^2 \leqslant \frac{I(p)}{I(p) + 4\sin^2\left(\frac{t}{2}\right)}, \qquad \forall\, t \in R. \tag{7}$$

Here, $I(p)$ is defined by equation (5).

Proof. We write the characteristic function $\phi(t)$ as the combination of its real part $\phi_{\Re}(t)$ and the imaginary part $\phi_{\Im}(t)$ as

$$\phi(t) = \phi_{\Re}(t) + \mathrm{i}\phi_{\Im}(t).$$

For any fixed $t \in R$, let us take

$$\mathbf{g}(x) = (\cos(tx), \sin(tx))$$

in the context of lemma 1, then the covariance matrix $\mathrm{Cov}(\mathbf{g}(X)) = (c_{ij})_{1\leqslant i,j\leqslant 2}$ can be straightforwardly evaluated as

$$\begin{aligned}
c_{11} &= E\cos^2(tX) - (E\cos(tX))^2\\
&= \tfrac{1}{2}(1 + \phi_{\Re}(2t)) - \phi_{\Re}^2(t);\\
c_{12} &= c_{21}\\
&= E(\cos(tX) - E\cos(tX))(\sin(tX) - E\sin(tX))\\
&= \tfrac{1}{2}\phi_{\Im}(2t) - \phi_{\Re}(t)\phi_{\Im}(t);\\
c_{22} &= E\sin^2(tX) - (E\sin(tX))^2\\
&= \tfrac{1}{2}(1 - \phi_{\Re}(2t)) - \phi_{\Im}^2(t).
\end{aligned}$$

If we put

$$A(t) = (\mathrm{e}^{\mathrm{i}t} - 1)\phi(t) = A_{\Re}(t) + \mathrm{i}A_{\Im}(t)$$

where $A_{\Re}(t)$ and $A_{\Im}(t)$ are the real and imaginary parts of $A(t)$, respectively, then from the identity

$$\cos(t(X+1)) - \cos(tX) = (\cos(t) - 1)\cos(tX) - \sin(t) \cdot \sin(tX)$$

we have

$$E(\cos(t(X+1)) - \cos(tX)) = (\cos(t) - 1)\phi_{\Re}(t) - \sin(t) \cdot \phi_{\Im}(t) = A_{\Re}(t).$$

Similarly, from

$$\sin(t(X+1)) - \sin(tX) = (\cos(t) - 1)\sin(tX) + \sin(t) \cdot \cos(tX)$$

we have

$$E(\sin(t(X+1)) - \sin(tX)) = (\cos(t) - 1)\phi_{\Im}(t) + \sin(t) \cdot \phi_{\Re}(t) = A_{\Im}(t).$$

Consequently,

$$(E(D\mathbf{g}(X))' E(D\mathbf{g}(X)) = \begin{pmatrix} A_{\Re}(t) \\ A_{\Im}(t) \end{pmatrix} (A_{\Re}(t), A_{\Im}(t)) = \begin{pmatrix} A_{\Re}^2(t) & A_{\Re}(t)A_{\Im}(t) \\ A_{\Re}(t)A_{\Im}(t) & A_{\Im}^2(t) \end{pmatrix}.$$

Now applying lemma 1, we have

$$\begin{pmatrix} \frac{1}{2}(1 + \phi_{\Re}(2t)) - \phi_{\Re}^2(t) & \frac{1}{2}\phi_{\Im}(2t) - \phi_{\Re}(t)\phi_{\Im}(t) \\ \frac{1}{2}\phi_{\Im}(2t) - \phi_{\Re}(t)\phi_{\Im}(t) & \frac{1}{2}(1 - \phi_{\Re}(2t)) - \phi_{\Im}^2(t) \end{pmatrix} \geqslant \frac{1}{I(p)} \begin{pmatrix} A_{\Re}^2(t) & A_{\Re}(t)A_{\Im}(t) \\ A_{\Re}(t)A_{\Im}(t) & A_{\Im}^2(t) \end{pmatrix}.$$

Taking the trace of the above inequality, we obtain

$$1 - |\phi(t)|^2 \geqslant \frac{1}{I(p)}|A(t)|^2.$$

But from the definition of $A(t)$, we know that

$$|A(t)|^2 = 4\sin^2\left(\frac{t}{2}\right) \cdot |\phi(t)|^2.$$

The desired result follows. □

Corollary 1. *For any* $0 < \delta < \pi$, *put*

$$\omega(\delta) = \sup_{\delta \leqslant |t| \leqslant \pi} |\phi(t)|.$$

Then,

$$\omega(\delta) \leqslant \sqrt{\frac{I(p)}{I(p) + 4\sin^2\left(\frac{\delta}{2}\right)}}.$$

Proof. This follows readily form theorem 1 by noting that $\sin^2(t/2)$ is an even function and is increasing on $[0, \pi]$. □

Corollary 2. *Let*

$$\tau = \int_0^{2\pi} |\phi(t)|^2 \, \mathrm{d}t$$

be the average of $|\phi(t)|^2$ *in the period interval* $[0, 2\pi]$. *Then,*

$$\tau \leqslant 2\pi\sqrt{\frac{I(p)}{I(p)+4}}. \tag{8}$$

Proof. Noting that

$$\sin^2\left(\frac{t}{2}\right) = \frac{1-\cos(t)}{2},$$

we have

$$\frac{I(p)}{I(p)+4\sin^2\left(\frac{t}{2}\right)} = \frac{I(p)}{I(p)+2}\cdot\frac{1}{1-\frac{2}{I(p)+2}\cos(t)}.$$

Now inequality (8) follows from integrating inequality (7) in theorem 1 and making use of the well-known integral

$$\int_0^{2\pi}\frac{1}{1+a\cos(t)}\,\mathrm{d}t = \frac{2\pi}{\sqrt{1-|a|^2}}, \qquad |a|<1.$$

□

Remark 1. If the probability distribution F is supported on a half of the integer lattice

$$Z\bigcap[s,\infty) = \{s, s+1, s+2, \ldots\}$$

where s is any integer which may be negative. Then with the convention $p_{s-1} = 0$ and by modifying $I(p)$,

$$I(p) = \sum_{k=s}^{\infty}\frac{(p_k - p_{k-1})^2}{p_k}.$$

All the above results still hold. This is relevant for our later applications when the energy observable is the number operator.

3. Implications for quantum dynamics

Let us illustrate some interesting physical implications of our results. Following physicist's terminology and considering the evolution of an arbitrary initial quantum state $|\psi\rangle$ (represented by a normalized wavefunction in $L^2(R)$) driven by a time-independent energy observable H (represented by a self-adjoint operator on $L^2(R)$), the evolving state $|\psi_t\rangle$ is determined by the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\psi_t\rangle = H|\psi_t\rangle, \qquad |\psi_0\rangle = |\psi\rangle,$$

where $\hbar$ is the Planck constant divided by 2π. Formally, the solution is given by $|\psi_t\rangle = \mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle$, and the survival amplitude at time t is defined as

$$\phi(t) = \langle\psi|\psi_t\rangle = \langle\psi|\,\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle, \qquad t\in R.$$

Now assume that the energy spectrum of H is Z or rather $\{0, 1, 2, \ldots\}$ (which is the case when H is the number operator of a harmonic oscillator). The complete set of energy eigenfunctions $\{|k\rangle\}$ of H has the properties that

$$H|k\rangle = k|k\rangle, \qquad \langle k|k\rangle = 1, \qquad \langle k|j\rangle = 0 \qquad \text{for} \quad k\neq j.$$

A quantum state $|\psi\rangle$ can be expanded in terms of the complete set $\{|k\rangle\}$ as

$$|\psi\rangle = \sum_k c_k|k\rangle$$

with $c_k = \langle k|\psi\rangle$. Then by spectral analysis,

$$\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle = \sum_k \mathrm{e}^{-\mathrm{i}tk/\hbar}c_k|k\rangle,$$

and by the Parseval theorem,

$$\phi(t) = \langle\psi|\,\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle = \sum_k \mathrm{e}^{-\mathrm{i}tk/\hbar}|c_k|^2.$$

Consequently, the survival amplitude can be expressed as the characteristic function of the state probability $p = \{p_k = |c_k|^2 : k \in Z\}$ in the energy representation if we replace t by $-t/\hbar$. By inequality (7), we conclude that

$$|\phi(t)|^2 \leqslant \frac{I(p)}{I(p) + 4\sin^2\left(\frac{t}{2\hbar}\right)}, \qquad \forall\, t \in R.$$

Moreover, if we follow [3] and define the *average lifetime* of the state in the period $[0, 2\pi\hbar]$ as

$$\tau = \int_0^{2\pi\hbar} |\phi(t)|^2\,\mathrm{d}t,$$

then we readily have

$$\tau \leqslant 2\pi\hbar\sqrt{\frac{I(p)}{I(p)+4}}$$

which shows that the average lifetime in a period is bounded by a simple functional of the Fisher information.

In summary, we have established a Cramér–Rao-type inequality for vector functions of integer-valued random variables. By use of this matrix inequality, we obtain an upper bound for the characteristic functions of lattice distributions, which has some interesting physical implications. In particular, it can be readily applied to estimate the survival probability and average lifetime of quantum states associated with quantum harmonic oscillators.

References

[1] Benedicks M 1975 An estimate of the modulus of the characteristic function of a lattice distribution with application to remainder term estimates in local limit theorems *Ann. Prob.* **3** 162–5

[2] Feller W 1971 *An Introduction to Probability and its Applications* vol 2 2nd edn (New York: Wiley)

[3] Gislason E A, Sabelli N H and Wood J W 1985 New forms of the time–energy uncertainty relation *Phys. Rev.* A **31** 2078–81

[4] Gnedenko B V and Kolmogorov A N 1954 *Limit Distributions for Sums of Independent Random Variables* (Reading, MA: Addison-Wesley)

[5] Johnstone I M and MacGibbon B 1987 Une mesure dínformation caractérisant la loi de Poisson *Séminaire de Probabilités XXI* (*Lecture Notes in Math.* vol 1247) (Berlin: Springer) pp 563–73

[6] Lukacs E 1970 *Characteristic Functions* 2nd edn (London: Griffin)

[7] Luo S 2001 A variation of the Heisenberg uncertainty relation involving an average *J. Phys. A: Math. Gen.* **34** 3289–91

[8] Luo S 2004 How fast can a quantum state evolve into a target state? *Physica* D **189** 1–7

[9] Luo S, Wang Z and Zhang Q 2002 An inequality for characteristic functions and its applications to uncertainty relations and the quantum Zeno effect *J. Phys. A: Math. Gen.* **35** 5935–41

[10] Luo S and Zhang Z 2004 Estimating the first zero of a characteristic function *C. R. Acad. Sci., Paris* I **338** 203–6

[11] Luo S and Zhang Z 2005 On decaying rate of quantum states *Lett. Math. Phys.* **71** 1–11

[12] Luo S and Zhang Z 2005 An extremal problem for Fourier transforms of probabilities *C. R. Acad. Sci., Paris* I **341** 293–6

[13] Mandelstam L and Tamm I 1945 The uncertainty relation between energy and time in non-relativistic quantum mechanics *J. Phys. (USSR)* **9** 249–54

[14] Misra B and Sudarshan E C G 1977 The Zeno's paradox in quantum theory *J. Math. Phys.* **18** 756–63